Indian Statistical Institute, Bangalore Centre
B.Math (Hons.) II Year, First Semester
Semestral Examination
Analysis III

Time: 3 Hours December 5, 2012 Instructor: T.S.S.R.K. Rao
Total Marks : 5x10=50

Please state correctly any theorem that you use. If an answer is an immediate consequence of a Theorem, that theorem needs to be proved.

1. Let f, g integrable function on $[0,1]$. Define $F : [0,1] \times [0,1] \to \mathbb{R}$ by $F(x,y) = f(x) + g(y)$. Show that F is an integrable function.

2. Let f be an integrable function on the unit cube $I = [0,1]\times[0,1]\times[0,1]$. Define $F(x,y,z) = \iiint\limits_{[0,x]\times[0,y]\times[0,z]} f(u,v,w)\ d(u,v,w)$. Show that F is a uniformly continuous function on I.

3. Let $F : [0,1] \times [0,1] \to \mathbb{R}$ be continuous such that $F(1,0) = -F(0,1)$ and assume that both F_x, F_{xy} exists and are continuous. Show that $\iint\limits_{[0,1]\times[0,1]} F_{xy} = F(1,1) + F(0,0)$.

4. Consider the following set in $[0,1]\times[0,1]\times[0,1]$. $G = \{(0,y,1) : y$ rational$\}$ $\cup\{(x,y,\frac{1}{q}) : x = p/q, p, q$ relatively prime y rational$\}$ $\cup\{(x,y,0) :$ x, y irrational$\}$. Show that G is a set of content zero.

5. Let $\{S_n\}_{n\geq 1}$ be a sequence of open connected sets in $\mathbb{R}^2$ such that $S_n \cap S_m \neq \phi$ for all $n \neq m$. Show that $S = \bigcup\limits_{n\geq 1} S_n$ is a connected set.

6. Show that $P : [0,1] \to \mathbb{R}^3$ defined by $P(t) = \frac{1}{2}(t+\sqrt{t^2+1}, \frac{1}{t+\sqrt{t^2+1}}, \sqrt{2}\log(\sqrt{t^2+1}))$ is a unit speed parametrization.

7. Let U be an open set in $\mathbb{R}^3$, $F = (f_1, f_2, f_3)$ be a smooth vector field on U and let $f : U \to \mathbb{R}$ be a function with continuous 2nd order derivatives. Derive the formulae, $\mathrm{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$,
$$\mathrm{curl}(\mathrm{curl}\ F) = \nabla(\mathrm{div}\ F) - \nabla^2(F).$$

8. Consider a circle of radius r in the YZ - plane with center at $(0,b,0)$. Show that the area of the Torus obtained by revolving this circle about the Z-axis is $4\pi^2 rb$.

9. Let $f \in C[0,1]$ be such that $\int\limits_0^1 f\, dx = 0$. Show that there is a sequence of polynomials $\{P_n\} \subset C[0,1]$ such that $\int\limits_0^1 P_n\, dx = 0$ for all n and $P_n \to f$ uniformly.

10. Let $\{f_n\}_{n\geq 1} \subset C[0,1]$ be such that $f_n(x) \leq f_{n+1}(x)$ for all $x \in [0,1]$ and for all n. Suppose $f_n(x) \to 0$ as $n \to \infty$ for all x. Show that there is a $M > 0$ such that $\sup\limits_{\substack{0\leq x\leq 1\\ n\geq 1}} |f_n(x)| \leq M$.